

# ALEXANDROFF MANIFOLDS AND HOMOGENEOUS CONTINUA

A. KARASSEV, V. TODOROV, AND V. VALOV

**ABSTRACT.** We prove the following result announced in [18]: Any homogeneous, metric  $ANR$ -continuum is a  $V_G^n$ -continuum provided  $\dim_G X = n \geq 1$  and  $\check{H}^n(X; G) \neq 0$ , where  $G$  is a principal ideal domain. This implies that any homogeneous  $n$ -dimensional metric  $ANR$ -continuum with  $\check{H}^n(X; G) \neq 0$  is a  $V^n$ -continuum in the sense of Alexandroff [1]. We also prove that any finite-dimensional homogeneous metric continuum  $X$ , satisfying  $\check{H}^n(X; G) \neq 0$  for some group  $G$  and  $n \geq 1$ , cannot be separated by a compactum  $K$  with  $\check{H}^{n-1}(K; G) = 0$  and  $\dim_G K \leq n - 1$ . This provides a partial answer to a question of Kallipoliti-Papasoglu [11] whether any two-dimensional homogeneous Peano continuum cannot be separated by arcs.

## 1. INTRODUCTION

Cantor manifolds and stronger versions of Cantor manifolds were introduced to describe some properties of Euclidean manifolds. According to Bing-Borsuk conjecture [2] that any homogeneous metric  $ANR$ -compactum of dimension  $n$  is an  $n$ -manifold, finite-dimensional homogeneous metric  $ANR$ -continua are supposed to share some properties with Euclidean manifolds. One of the first results in that direction established by Krupski [15] is that any homogeneous metric continuum of dimension  $n$  is a Cantor  $n$ -manifold. Recall that a space  $X$  is a *Cantor  $n$ -manifold* if any partition of  $X$  is of dimension  $\geq n - 1$  [20] (a partition of  $X$  is a closed set  $P \subset X$  such that  $X \setminus P$  is the union of two open disjoint sets). In other words,  $X$  cannot be the union of two proper closed sets whose intersection is of covering dimension  $\leq n - 2$ . Stronger versions of Cantor manifolds were considered by Hadžiivanov [9] and Hadžiivanov and Todorov [10]. But the strongest specification

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of Cantor manifolds is the notion of  $V^n$ -continua introduced by Alexandroff [1]: a compactum  $X$  is a  $V^n$ -continuum if for every two closed disjoint massive subsets  $X_0, X_1$  of  $X$  there exists an open cover  $\omega$  of  $X$  such that there is no partition  $P$  in  $X$  between  $X_0$  and  $X_1$  admitting an  $\omega$ -map into a space  $Y$  with  $\dim Y \leq n - 2$  ( $f: P \rightarrow Y$  is said to be an  $\omega$ -map if there exists an open cover  $\gamma$  of  $Y$  such that  $f^{-1}(\gamma)$  refines  $\omega$ ). Recall that a massive subset of  $X$  is a set with non-empty interior in  $X$ .

More general concepts of the above notions were considered in [13]. In particular, we are going to use the following one, where  $\mathcal{C}$  is a class of topological spaces.

**Definition 1.1.** A space  $X$  is an *Alexandroff manifold with respect to  $\mathcal{C}$*  (br., *Alexandroff  $\mathcal{C}$ -manifold*) if for every two closed, disjoint, massive subsets  $X_0, X_1$  of  $X$  there exists an open cover  $\omega$  of  $X$  such that there is no partition  $P$  in  $X$  between  $X_0$  and  $X_1$  admitting an  $\omega$ -map onto a space  $Y \in \mathcal{C}$ .

In this paper we continue investigating to what extent homogeneous continua have common properties with Euclidean manifolds. One of the main questions in this direction is whether any homogeneous  $n$ -dimensional metric *ANR*-compactum  $X$  is a  $V^n$ -continuum, see [18]. A partial answer of this question, when Čech cohomology group  $\check{H}^n(X)$  is non-trivial, was announced in [18]. One of the aims of the paper is to provide the proof of this fact, see Section 3. Our proof is based on the properties of  $(n, G)$ -bubbles and  $V_G^n$ -continua investigated in Section 2. We also provide a partial answer to a question of Kallipoliti-Papasoglu [11].

## 2. $(n, G)$ -BUBBLES AND $V_G^n$ -CONTINUA

In this section we investigate the connection between  $(n, G)$ -bubbles and  $V_G^n$ -continua.

For every abelian group  $G$  let  $\dim_G X$  be the cohomological dimension of  $X$  with respect to  $G$ , and  $\check{H}^n(X; G)$  denotes the reduced  $n$ -th Čech cohomology group of  $X$  with coefficients in  $G$ .

Reformulating the original definition of Kuperberg [16], Yokoi [21] provided the following definition (see also [3] and [12]):

**Definition 2.1.** If  $G$  is an abelian group and  $n \geq 0$ , a compactum  $X$  is called an  $(n, G)$ -bubble if  $\check{H}^n(X; G) \neq 0$  and  $\check{H}^n(A; G) = 0$  for every proper closed subset  $A$  of  $X$ . Following [19] we say that a compactum  $X$  is a *generalized  $(n, G)$ -bubble* provided there exists a surjective map  $f: X \rightarrow Y$  such that the homomorphism  $f^*: \check{H}^n(Y; G) \rightarrow \check{H}^n(X; G)$

is nontrivial, but  $f_A^*(\check{H}^n(Y; G)) = 0$  for any proper closed subset  $A$  of  $X$ , where  $f_A$  is the restriction of  $f$  over  $A$ .

We also need the following notion:

**Definition 2.2.** A compactum  $X$  is said to be a  $V_G^n$ -continuum [17] if for every two closed, disjoint, massive subsets  $X_0, X_1$  of  $X$  there exists an open cover  $\omega$  of  $X$  such that any partition  $P$  in  $X$  between  $X_0$  and  $X_1$  does not admit an  $\omega$ -map  $g$  onto a space  $Y$  with  $g^*: \check{H}^{n-1}(Y; G) \rightarrow \check{H}^{n-1}(P; G)$  being trivial.

Since  $\check{H}^{n-1}(Y; G) = 0$  for any compactum  $Y$  with  $\dim_G Y \leq n - 2$ ,  $V_G^n$ -continua are Alexandroff manifolds with respect to the class  $D_G^{n-2}$  of all spaces of dimension  $\dim_G \leq n - 2$ . Moreover, if  $X \in V_G^n$ , then for every partition  $C$  of  $X$  we have  $\check{H}^{n-1}(C; G) \neq 0$ . The last observation implies  $\dim_G X \geq n$  provided  $X$  is a metric  $V_G^n$ -compactum such that either  $X \in ANR$  or  $\dim X < \infty$  and  $G$  is countable. Indeed, if  $\dim_G X \leq n - 1$ , then each  $x \in X$  has a local base of open sets  $U$  whose boundaries are of dimension  $\dim_G \leq n - 2$ , see [7]. Hence, any such a boundary  $\Gamma$  is a partition of  $X$  with  $\check{H}^{n-1}(\Gamma; G) = 0$ .

Next theorem was established in [17, Theorem 3] for finite-dimensional metric  $(n, G)$ -bubbles. Let us note that, according to [21], the examples of Dranishnikov [4] and Dydak-Walsh [6] show the existence of an infinite-dimensional  $(n, \mathbb{Z})$ -bubble with  $n \geq 5$ .

**Theorem 2.3.** *Any generalized  $(n, G)$ -bubble  $X$  is a  $V_G^n$ -continuum.*

*Proof.* Let  $f: X \rightarrow Y$  be a map such that  $f^*(\check{H}^n(Y; G)) \neq 0$  and  $f_A^*(\check{H}^n(Y; G)) = 0$  for any proper closed set  $A \subset X$ . If  $\omega$  is a finite open cover of a closed set  $Z \subset X$ , we denote by  $|\omega|$  and  $p_\omega$ , respectively, the nerve of  $\omega$  and a map from  $Z$  onto  $|\omega|$  generated by a partition of unity subordinated to  $\omega$ . Furthermore, if  $C \subset Z$  and  $\omega(C) = \{W \in \omega : W \cap C \neq \emptyset\}$ , then  $p_{\omega(C)}: C \rightarrow |\omega(C)|$  is the restriction  $p_\omega|_C$ . Recall also that  $p_\omega$  generates maps  $p_\omega^*: \check{H}^k(|\omega|; G) \rightarrow \check{H}^k(Z; G)$ ,  $k \geq 0$ . Moreover, if  $q_\omega: Z \rightarrow |\omega|$  is a map generating by (another) partition of unity subordinated to  $|\omega|$ , then  $p_\omega$  and  $q_\omega$  are homotopic. So,  $p_\omega^* = q_\omega^*$ .

*Claim 1.* *For every non-empty open sets  $U_1$  and  $U_2$  in  $X$  with  $\overline{U_1} \cap \overline{U_2} = \emptyset$  there exist an open cover  $\omega$  of  $X \setminus (U_1 \cup U_2)$ , a map  $p_\omega: X \setminus (U_1 \cup U_2) \rightarrow |\omega|$  and an element  $e \in \check{H}^{n-1}(|\omega|; G)$  such that  $p_{\omega(C)}^*(i_C^*(e)) \neq 0$  for every partition  $C$  of  $X$  between  $\overline{U_1}$  and  $\overline{U_2}$ , where  $i_C$  is the inclusion  $|\omega(C)| \hookrightarrow |\omega|$ .*

To prove this claim we follow the arguments from the proof of [19, Theorem]. Let  $U_1$  and  $U_2$  be non-empty open subsets of  $X$  with disjoint

closures, and  $i_k : F_k \hookrightarrow X$  be the inclusion of  $F_k = X \setminus U_k$  into  $X$ ,  $k = 1, 2$ . Consider the Mayer-Vietoris exact sequence

$$\check{H}^{n-1}(F_1 \cap F_2; G) \xrightarrow{\delta} \check{H}^n(X; G) \xrightarrow{j} \check{H}^n(F_1; G) \oplus \check{H}^n(F_2; G) \rightarrow$$

with  $j = (i_1^*, i_2^*)$ , and choose a non-zero element  $e_1 \in f^*(\check{H}^n(Y; G)) \subset \check{H}^n(X; G)$ . For each  $k = 1, 2$  we have the commutative diagram, where  $\delta_k$  is the inclusion of  $f(F_k)$  into  $Y$ :

$$\begin{array}{ccc} \check{H}^n(Y; G) & \xrightarrow{f^*} & \check{H}^n(X; G) \\ \downarrow \delta_k^* & & \downarrow i_k^* \\ \check{H}^n(f(F_k); G) & \xrightarrow{f_{F_k}^*} & \check{H}^n(F_k; G). \end{array}$$

So  $i_k^*(e_1) = 0$ ,  $k = 1, 2$ , which yields  $e_1 = \delta(e_2)$  for some non-zero element  $e_2 \in \check{H}^{n-1}(F_1 \cap F_2; G)$ . Then there exist an open cover  $\omega$  of  $F_1 \cap F_2 = X \setminus (U_1 \cup U_2)$ , a map  $p_\omega : F_1 \cap F_2 \rightarrow |\omega|$  and  $e \in \check{H}^{n-1}(|\omega|; G)$  with  $p_\omega^*(e) = e_2$ .

Let  $C$  be a partition of  $X$  between  $\overline{U}_1$  and  $\overline{U}_2$ . So,  $X = P_1 \cup P_2$  and  $C = P_1 \cap P_2$ , where each  $P_k$  is a closed subset of  $X$  containing  $\overline{U}_k$ ,  $k = 1, 2$ . Denote by  $i : C \hookrightarrow F_1 \cap F_2$ ,  $i_1 : P_1 \hookrightarrow F_2$  and  $i_2 : P_2 \hookrightarrow F_1$  the corresponding inclusions. Then we have the following commutative diagram, whose rows are Mayer-Vietoris sequences:

$$\begin{array}{ccccccc} \check{H}^{n-1}(F_1 \cap F_2; G) & \xrightarrow{\delta} & \check{H}^n(X; G) & \xrightarrow{j} & \check{H}^n(F_2; G) \oplus \check{H}^n(F_1; G) \\ \downarrow i^* & & \downarrow id & & \downarrow i_1^* \oplus i_2^* \\ \check{H}^{n-1}(C; G) & \xrightarrow{\delta_1} & \check{H}^n(X; G) & \xrightarrow{j_1} & \check{H}^n(P_1; G) \oplus \check{H}^n(P_2; G). \end{array}$$

Obviously,

$$(1) \quad \delta_1(i^*(e_2)) = id(\delta(e_2)) = e_1 \neq 0.$$

On the other hand, the commutativity of the diagram

$$\begin{array}{ccc} \check{H}^{n-1}(|\omega|; G) & \xrightarrow{p_\omega^*} & \check{H}^{n-1}(F_1 \cap F_2; G) \\ \downarrow i_C^* & & \downarrow i^* \\ \check{H}^{n-1}(|\omega(C)|; G) & \xrightarrow{p_{\omega(C)}^*} & \check{H}^{n-1}(C; G) \end{array}$$

implies that  $p_{\omega(C)}^*(i_C^*(e)) = i^*(p_\omega^*(e)) = i^*(e_2)$ . Therefore, according to (1),  $p_{\omega(C)}^*(i_C^*(e)) \neq 0$ . This completes the proof of Claim 1.

Now, we can show that  $X \in V_G^n$ . Let  $U_1$  and  $U_2$  be non-empty open subsets of  $X$  with disjoint closures. Then there exists a finite open cover  $\omega$  of  $X \setminus (U_1 \cup U_2)$ , a map  $p_\omega: X \setminus (U_1 \cup U_2) \rightarrow |\omega|$  and an element  $e \in \check{H}^{n-1}(|\omega|; G)$  satisfying the conditions from Claim 1. For each  $W \in \omega$  let  $h(W)$  be an open subset of  $X$  extending  $W$ . So,  $\gamma = \{h(W) : W \in \omega\} \cup \{U_1, U_2\}$  is a finite open cover of  $X$  whose restriction on  $X \setminus (U_1 \cup U_2)$  is  $\omega$ .

Suppose there exists a partition  $C$  of  $X$  between  $\overline{U}_1$  and  $\overline{U}_2$  admitting a  $\gamma$ -map  $g$  onto a space  $T$  with  $g^*(\check{H}^{n-1}(T; G)) = 0$ . Thus, we can find a finite open cover  $\alpha$  of  $T$  such that  $\beta = g^{-1}(\alpha)$  is refining  $\omega$ . Let  $p_\beta: C \rightarrow |\beta|$  be a map onto the nerve of  $\beta$  generated by a partition of unity subordinated to  $\beta$ . Obviously, the function  $V \in \alpha \rightarrow g^{-1}(V) \in \beta$  generates a simplicial homeomorphism  $g_\beta^*: |\alpha| \rightarrow |\beta|$ . Then the maps  $p_\beta$  and  $g_\alpha = g_\beta^* \circ \pi_\alpha \circ g$ , where  $\pi_\alpha: T \rightarrow |\alpha|$  is a map generated by a partition of unity subordinated to  $\alpha$ , are homotopic. Hence,  $p_\beta^* = g^* \circ \pi_\alpha^* \circ (g_\beta^*)^*$ . Because  $g^*: \check{H}^{n-1}(T; G) \rightarrow \check{H}^{n-1}(C; G)$  is a trivial map, the last equality implies that so is the map  $p_\beta^*: \check{H}^{n-1}(|\beta|; G) \rightarrow \check{H}^{n-1}(C; G)$ . On the other hand, since  $\beta$  refines  $\omega$ , we can find a map  $\varphi_\beta: |\beta| \rightarrow |\omega(C)|$  such that  $p_{\omega(C)}$  and  $\varphi_\beta \circ p_\beta$  are homotopic. Therefore,  $p_{\omega(C)}^* = p_\beta^* \circ \varphi_\beta^*$ . According to Claim 1,  $p_{\omega(C)}^*(e_C) \neq 0$ , where  $e_C$  is the element  $i_C^*(e) \in \check{H}^{n-1}(|\omega(C)|; G)$ . So,  $p_\beta^*(\varphi_\beta^*(e_C)) \neq 0$ , which contradicts the triviality of  $p_\beta^*$ .  $\square$

We can extend the definition of  $V_G^n$ -continua as follows:

**Definition 2.4.** A compactum  $X$  is said to be a  $V_G^n$ -continuum with respect to a given class  $\mathcal{A}$  if for every two closed, disjoint, massive subsets  $X_0, X_1$  of  $X$  there exists an open cover  $\omega$  of  $X$  such that any partition  $P$  in  $X$  between  $X_0$  and  $X_1$  does not admit an  $\omega$ -map  $g$  onto a space  $Y \in \mathcal{A}$  with  $g^*: \check{H}^{n-1}(Y; G) \rightarrow \check{H}^{n-1}(P; G)$  being trivial.

Recall that a metric space  $X$  is *strongly  $n$ -universal* if any map  $g: K \rightarrow X$ , where  $K$  is a metric compactum of dimension  $\dim K \leq n$ , can be approximated by embeddings.

**Theorem 2.5.** *Let  $X$  be a metric compactum containing a strongly  $n$ -universal dense subspace  $M$  such that  $M$  is an absolute extensor for  $n$ -dimensional compacta with  $n \geq 1$ . Then  $X$  is a  $V_G^n$ -continuum with respect to the class  $D_G^{n-1}$  of all spaces of dimension  $\dim_G \leq n - 1$ . In particular,  $X$  is an Alexandroff manifold with respect to the class  $D_G^{n-2}$ .*

*Proof.* Suppose that  $X$  is not a  $V_G^n$ -continuum with respect to the class  $D_G^{n-1}$ . So, we can find open sets  $U$  and  $V$  in  $X$  with disjoint closures such that for every  $\epsilon > 0$  there exists a partition  $C_\epsilon$  between

$\overline{U}$  and  $\overline{V}$  admitting an  $\epsilon$ -map  $g_\epsilon$  onto a space  $Y_\epsilon \in D_G^{n-1}$  such that  $g_\epsilon^*: \check{H}^{n-1}(Y_\epsilon; G) \rightarrow \check{H}^{n-1}(C_\epsilon; G)$  is trivial.

Consider two different points  $a, b$  from the  $n$ -sphere  $\mathbb{S}^n$ , and a map  $f: \mathbb{S}^n \rightarrow M$  with  $f(a) \in U \cap M$  and  $f(b) \in V \cap M$  (such a map exists because  $M$  is an absolute extensor for  $n$ -dimensional compacta). Since  $M$  is strongly  $n$ -universal, we can approximate  $f$  by a homeomorphism  $h: \mathbb{S}^n \rightarrow M$  such that  $h(a) \in U$  and  $h(b) \in V$ . Therefore,  $K_\epsilon = C_\epsilon \cap h(\mathbb{S}^n)$  is a partition of  $h(\mathbb{S}^n)$  between  $h(\mathbb{S}^n) \cap \overline{U}$  and  $h(\mathbb{S}^n) \cap \overline{V}$ . Then  $Z = g_\epsilon(K_\epsilon)$  is a closed subset of  $Y_\epsilon$ , and since  $\dim_G Y_\epsilon \leq n-1$ ,  $i_Z^*: \check{H}^{n-1}(Y_\epsilon; G) \rightarrow \check{H}^{n-1}(Z; G)$  is a surjective map, where  $i_Z: Z \hookrightarrow Y_\epsilon$  is the inclusion. So, we have the following commutative diagram with  $g_{K_\epsilon} = g|_{K_\epsilon}$  and  $i_{K_\epsilon}: K_\epsilon \hookrightarrow C_\epsilon$ :

$$\begin{array}{ccc} \check{H}^{n-1}(Y_\epsilon; G) & \xrightarrow{g_\epsilon^*} & \check{H}^{n-1}(C_\epsilon; G) \\ \downarrow i_Z^* & & \downarrow i_{K_\epsilon}^* \\ \check{H}^{n-1}(Z; G) & \xrightarrow{g_{K_\epsilon}^*} & \check{H}^{n-1}(K_\epsilon; G). \end{array}$$

Because  $g_\epsilon^*$  is trivial and  $i_Z^*$  is surjective,  $g_{K_\epsilon}^*$  is also trivial. Hence, for every  $\epsilon > 0$  there exists a partition  $K_\epsilon$  between  $h(\mathbb{S}^n) \cap \overline{U}$  and  $h(\mathbb{S}^n) \cap \overline{V}$  admitting an  $\epsilon$ -map  $g_{K_\epsilon}$  onto a space  $Z$  such that  $g_{K_\epsilon}^*: \check{H}^{n-1}(Z; G) \rightarrow \check{H}^{n-1}(K_\epsilon; G)$  is trivial. This means that  $\mathbb{S}^n$  is not a  $V_G^n$ -continuum. On the other hand,  $\mathbb{S}^n$  is an  $(n, G)$ -bubble for all  $G$ . So, by Theorem 2.3,  $\mathbb{S}^n$  is a  $V_G^n$ -continuum - a contradiction.  $\square$

**Corollary 2.6.** *Let  $X$  be either the universal Menger compactum  $\mu^n$  or  $X$  be a metric compactification of the universal Nöbeling space  $\nu^n$ . Then  $X$  is a  $V_G^n$ -continuum with respect to the class  $D_G^{n-1}$  for any  $G$ . Moreover,  $\mu^n$  is not a  $V_G^n$ -continuum.*

*Proof.* Since both  $\mu^n$  and  $\nu^n$  are strongly  $n$ -universal absolute extensors for  $n$ -dimensional compacta, it follows from Theorem 2.5 that  $X$  is a  $V_G^n$ -continuum with respect to the class  $D_G^{n-1}$ . To show that  $\mu^n$  is not a  $V_G^n$ -continuum, it suffices to find a partition  $E$  of  $\mu^n$  with trivial  $\check{H}^{n-1}(E; G)$ . One can show the existence of such partitions using the geometric construction of the Menger compactum. We provide a proof of this fact using Dranishnikov's results from [5]. Indeed, by [5, Theorem 2], there exists a map  $g: \mu^n \rightarrow \mathbb{I}^\infty$  such that  $g^{-1}(P)$  is homeomorphic to  $\mu^n$  for any  $AR$ -space  $P \subset \mathbb{I}^\infty$ . If  $P \in AR$  is a partition of  $\mathbb{I}^\infty$ , then  $g^{-1}(P)$  is a partition of  $\mu^n$  homeomorphic to  $\mu^n$ . Hence,  $\check{H}^{n-1}(g^{-1}(P); G) = 0$ .  $\square$

### 3. HOMOGENEOUS CONTINUA

In this section we prove that some homogeneous continua are  $V_G^n$ -continua. Recall that a space  $X$  is said to be *homogeneous* if for every two points  $x, y \in X$  there exist a homeomorphism  $h : X \rightarrow X$  with  $h(x) = y$ . Krupski [14] conjectured that any  $n$ -dimensional, homogeneous metric  $ANR$ -continuum is a  $V^n$ -continuum. Next result provides a partial solution to Krupski's conjecture and a partial answer to Question 2.4 from [18].

**Theorem 3.1.** *Let  $X$  be a homogeneous, metric  $ANR$ -continuum with  $\dim_G X = n \geq 1$  such that  $\check{H}^n(X; G) \neq 0$ , where  $G$  is a principal ideal domain. Then  $X$  is a  $V_G^n$ -continuum.*

*Proof.* According to [21, Theorem 3.3], any space  $X$  satisfying the conditions from this theorem is an  $(n, G)$ -bubble. Hence, by Theorem 2.3,  $X \in V_G^n$ .  $\square$

Bing and Borsuk [2] raised the question whether no compact acyclic in dimension  $n - 1$  subset of  $X$  separates  $X$ , where  $X$  is a metric  $n$ -dimensional homogeneous  $ANR$ -continuum. Yokoi [21, Corollary 3.4] provided a partial positive answer to this question in the case  $X$  is a homogeneous metric  $n$ -dimensional  $ANR$ -continuum such that  $\check{H}^n(X; \mathbb{Z}) \neq 0$ . Next proposition is a version of Yokoi's result when  $X$  is not necessarily  $ANR$ .

**Proposition 3.2.** *Let  $X$  be a finite-dimensional homogeneous metric continuum with  $\check{H}^n(X; G) \neq 0$ . Then  $\check{H}^{n-1}(C; G) \neq 0$  for any partition  $C$  of  $X$  such that  $\dim_G C \leq n - 1$ .*

*Proof.* Suppose there exists a partition  $C$  of  $X$  such that  $\check{H}^{n-1}(C; G) = 0$  and  $\dim_G C \leq n - 1$ . The last inequality implies that the inclusion homomorphism  $\check{H}^{n-1}(C; G) \rightarrow \check{H}^{n-1}(A; G)$  is an epimorphism for every closed set  $A \subset C$ . So,  $\check{H}^{n-1}(A; G) = 0$  for all closed subsets of  $C$ . Therefore, we may assume that  $C$  does not have any interior points. Since  $\check{H}^n(X; G) \neq 0$ , according to [17, Theorem 2], there exists a compact subset  $K \subset X$  with  $K \in V_G^n$ . Since  $X$  is homogeneous, we may also assume that  $K \cap C \neq \emptyset$ . Observe that  $z \in K \setminus C$  for some  $z$ . Indeed, the inclusion  $K \subset C$  would imply that  $\check{H}^{n-1}(P; G) = 0$  for every partition  $P$  of  $K$ . Let  $X \setminus C = U \cup V$  and  $z \in V$ , where  $U$  and  $V$  are nonempty, open and disjoint sets in  $X$ . Then the Effros theorem [8] allows us to push  $K$  towards  $U$  by a small homeomorphism  $h : X \rightarrow X$  so that the image  $h(K)$  meets both  $U$  and  $V$  (see the proof of Lemma 2 from [15] for a similar application of Effros' theorem). Therefore,

$S = h(K) \cap C$  is a partition of  $h(K)$  and  $\check{H}^{n-1}(S; G) = 0$  because  $S \subset C$ , a contradiction.  $\square$

Proposition 3.2 provides a partial answer to a question of Kallipoliti-Papasoglu [11] whether homogeneous two-dimensional metric locally connected continua cannot be separated by arcs.

**Corollary 3.3.** *No finite-dimensional metric homogeneous continuum  $X$  with  $\check{H}^2(X; G) \neq 0$  can be separated by any one-dimensional compactum  $C$  with  $\check{H}^1(C; G) = 0$ .*

#### 4. SOME REMARKS AND PROBLEMS

The class of  $(n, G)$ -bubbles is stable in the sense of the following proposition.

**Proposition 4.1.** *Let  $X$  be a metric compactum admitting an  $\epsilon$ -map onto an  $(n, G)$ -bubble for any  $\epsilon > 0$ . Then  $X$  is also an  $(n, G)$ -bubble.*

*Proof.* First, let us show that  $\check{H}^n(X; G) \neq 0$ . Take any open cover  $\omega$  of  $X$  and let  $\epsilon$  be the Lebesgue number of  $\omega$ . There exists a surjective  $\epsilon$ -map  $f: X \rightarrow Y_\epsilon$  with  $Y_\epsilon$  being an  $(n, G)$ -bubble. Since  $\check{H}^n(Y_\epsilon; G) \neq 0$ , we can find an open cover  $\alpha$  of  $Y_\epsilon$  such that  $\check{H}^n(|\alpha|; G) \neq 0$  (we use the notations from the proof of Theorem 2.3). Then  $\beta = f^{-1}(\alpha)$  is an open cover of  $X$  refining  $\omega$  such that  $|\beta|$  is homeomorphic to  $|\alpha|$ . So,  $\check{H}^n(|\beta|; G) \neq 0$ , which implies  $\check{H}^n(X; G) \neq 0$ .

Suppose now that  $A$  is a proper closed subset of  $X$  and  $\gamma$  an open (in  $A$ ) cover of  $A$ . Extend each  $U \in \gamma$  to an open set  $V(U)$  in  $X$  and let  $W = \cup\{V(U) : U \in \gamma\}$ . We can suppose that  $W \neq X$ . Choose a surjective  $\eta$ -map  $g: X \rightarrow Y_\eta$  such that  $Y_\eta$  is an  $(n, G)$ -bubble with  $\eta$  being a positive number smaller than both  $\text{dist}(A, X \setminus W)$  and the Lebesgue number of  $\gamma$ . Then  $B = g(A)$  is a proper closed subset of  $Y_\eta$  such that  $g^{-1}(B) \subset W$ . There exists an open cover  $\theta$  of  $B$  such that the family  $\delta = \{g^{-1}(G) \cap A : G \in \theta\}$  is an open cover of  $A$  refining  $\gamma$ . Obviously,  $|\delta|$  is homeomorphic to  $|\theta|$ . Since  $\check{H}^n(B; G) = 0$ , we have  $\check{H}^n(|\theta|; G) = \check{H}^n(|\delta|; G) = 0$ . Hence  $\check{H}^n(A; G) = 0$ , which completes the proof.  $\square$

Now, we are going to discuss some problems. The main question suggested by the results from this paper is to remove in Theorem 3.1 some of the conditions about  $X$ . Since, according to Corollary 2.3,  $\mu^n$  is not a  $V_G^n$ -continuum for any  $G$ , the condition  $X$  to be an ANR cannot be removed. So, we have the following question.

**Problem 4.1.** *Let  $X$  be a homogeneous metric ANR-continuum  $X$  with  $\dim_G X = n$ , where  $G$  is any abelian group. Is  $X$  a  $V_G^n$ -continuum?*



Since any  $V_G^n$ -continuum with respect to the class  $D_G^{n-1}$  is  $V^n$ , next question is still interesting.

**Problem 4.2.** *Let  $X$  be a homogeneous metric continuum  $X$  with  $\dim_G X = n$ . Is  $X$  a  $V_G^n$ -continuum with respect to the class  $D_G^{n-1}$ ? What if  $\check{H}^n(X; G) \neq 0$ ?*

Another question is whether finite-dimensionality can be removed from the result of Stefanov [17] which was applied above.

**Problem 4.3.** *Let  $X$  be a metrizable compactum with  $\check{H}^n(X; G) \neq 0$  for some group  $G$  and  $n \geq 1$ . Does  $X$  contain a  $V_G^n$ -continuum?*

We can show that any finite simplicial complex is a generalized  $(n, G)$ -bubble if and only if it is an  $(n, G)$ -bubble. So, our last question is whether this remain true for all metric compacta.

**Problem 4.4.** *Is there any metric compactum  $X$  which is a generalized  $(n, G)$ -bubble but not an  $(n, G)$ -bubble?*

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DEPARTMENT OF COMPUTER SCIENCE AND MATHEMATICS, NIPISSING UNIVERSITY, 100 COLLEGE DRIVE, P.O. Box 5002, NORTH BAY, ON, P1B 8L7, CANADA

*E-mail address:* alexandk@nipissingu.ca

DEPARTMENT OF MATHEMATICS, UACG, 1 H. SMIRNENSKI BLVD., 1046 SOFIA, BULGARIA

*E-mail address:* vtt-fte@uacg.bg

DEPARTMENT OF COMPUTER SCIENCE AND MATHEMATICS, NIPISSING UNIVERSITY, 100 COLLEGE DRIVE, P.O. Box 5002, NORTH BAY, ON, P1B 8L7, CANADA

*E-mail address:* veskov@nipissingu.ca